

ON THE KÄHLER METRICS OVER $\mathrm{Sym}^d(X)$

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ABSTRACT. Let X be a compact connected Riemann surface of genus g , with $g \geq 2$. For each $d < \eta(X)$, where $\eta(X)$ is the gonality of X , the symmetric product $\mathrm{Sym}^d(X)$ embeds into $\mathrm{Pic}^d(X)$ by sending an effective divisor of degree d to the corresponding holomorphic line bundle. Therefore, the restriction of the flat Kähler metric on $\mathrm{Pic}^d(X)$ is a Kähler metric on $\mathrm{Sym}^d(X)$. We investigate this Kähler metric on $\mathrm{Sym}^d(X)$. In particular, we estimate its Bergman kernel. We also prove that any holomorphic automorphism of $\mathrm{Sym}^d(X)$ is an isometry.

1. INTRODUCTION

Symmetric products of Riemann surfaces were studied by Macdonald [10]; he explicitly computed their cohomologies. Interests on these varieties revived when it was realized that they constitute examples of vortex moduli spaces [3], [4], [7]. One of the questions was to compute the volume, which was resolved in a series of papers [13], [11], [14]; see also [1] for Kähler structure on vortex moduli spaces.

Let X be a compact connected Riemann surface of genus g , with $g \geq 2$, and let $\eta(X)$ denote the gonality of X (this means that X admits a nonconstant holomorphic map to \mathbb{CP}^1 of degree $\eta(X)$ and it does not have any smaller degree nonconstant holomorphic map to \mathbb{CP}^1). Take any integer $1 \leq d < \eta(X)$. Let

$$\varphi : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^d(X)$$

be the map from the symmetric product that sends any $\{x_1, \dots, x_d\}$ to the holomorphic line bundle $\mathcal{O}_X(x_1 + \dots + x_d)$. We prove that φ is an embedding.

The natural inner product on $H^0(X, K_X)$, where $K_X \rightarrow X$ is the holomorphic cotangent bundle, produces a flat Kähler metric on $\mathrm{Pic}^d(X)$. It is natural to construct a metric on $\mathrm{Sym}^d(X)$ by pulling back the flat metric using the embedding φ ; see [15], [12] (especially [12, p. 1137, (1.2)], [12, § 7]). Our aim here is to study this metric on $\mathrm{Sym}^d(X)$. We prove that any holomorphic automorphism of $\mathrm{Sym}^d(X)$ is in fact an isometry. Our main result is estimation of the Bergman kernel of the metric.

Classically, the Bergman kernel which is the reproducing kernel for L^2 -holomorphic functions has been extensively studied in complex analysis. The generalization of the Bergman kernel to complex manifolds as the kernel for the projection onto the space of harmonic (p, q) -forms with L^2 -coefficients carries the information on the algebraic and geometric structures of the underlying manifolds.

2010 *Mathematics Subject Classification.* 14H40, 14H81, 53C07.

Key words and phrases. Symmetric product; Jacobian; Bergman kernel, Petersson inner product.

Using results from [6] and [9], we derive the following estimate for $\mathcal{B}_X(z)$, the Bergman kernel associated to the Riemann surface X :

$$\mathcal{B}_X(z) \leq \frac{48}{\pi} + \frac{4}{3\pi \sinh^2(r_X/4)},$$

where r_X denotes the injectivity radius of X .

We also study the above estimate for admissible sequences of compact hyperbolic Riemann surfaces. Our estimates are optimal, and these estimates continue to hold true for any compact hyperbolic Riemann surface.

2. COMPARISON OF KÄHLER METRICS

In this section, we introduce the hyperbolic and canonical metrics defined on a compact hyperbolic Riemann surface. Furthermore, we introduce the Bergman kernel, and derive estimates for it. We then extend these estimates to admissible sequences of compact hyperbolic Riemann surfaces.

2.1. Canonical and hyperbolic metrics. Let X be a compact, connected Riemann surface of genus g , with $g > 1$. Let

$$\mathbb{H} := \{z = x + \sqrt{-1}y \in \mathbb{C} \mid y > 0\}$$

be the upper half-plane. Using the uniformization theorem X can be realized as the quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a torsionfree cocompact Fuchsian subgroup acting on \mathbb{H} , via fractional linear transformations.

Locally, we identify X with its universal cover \mathbb{H} using the covering map $\mathbb{H} \rightarrow X$.

The holomorphic cotangent bundle on X will be denoted by K_X . Let

$$\mathrm{Jac}(X) = \mathrm{Pic}^0(X)$$

be the Jacobian variety that parametrizes all the (holomorphic) isomorphism classes of topologically trivial holomorphic line bundles on X . It is equipped with a flat Kähler metric g_J given by the Hermitian structure on $H^0(X, K_X)$ defined by

$$(\alpha, \beta) \mapsto \frac{\sqrt{-1}}{2} \int_X \alpha \wedge \bar{\beta}. \quad (2.1)$$

Fix a base point $x_0 \in X$. Let

$$\mathrm{AJ}_X : X \rightarrow \mathrm{Jac}(X)$$

be the Abel-Jacobi map that sends any $x \in X$ to the holomorphic line bundle on X of degree zero given by the divisor $x - x_0$. It is a holomorphic embedding of X . The pulled back Kähler metric $\mathrm{AJ}_X^* g_J$ on X is called the *canonical metric*. The $(1,1)$ -form on X associated to the canonical metric is denoted by μ_X^{can} .

The canonical metric has the following alternate description. Let $S_2(\Gamma)$ denote the \mathbb{C} -vector space of cusp forms of weight-2 with respect to Γ . Let $\{f_1, \dots, f_g\}$ denote an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. Then, the $(1,1)$ -form $\mu_X^{\mathrm{can}}(z)$ corresponding to the canonical metric of X is given by

$$\mu_X^{\mathrm{can}}(z) := \frac{\sqrt{-1}}{2g} \sum_{j=1}^g |f_j(z)|^2 dz \wedge d\bar{z}. \quad (2.2)$$

The volume of X with respect to the canonical metric is one.

Consider the hyperbolic metric of X , which is compatible with the complex structure on X and has constant negative curvature -1 . We denote by μ_X^{hyp} the $(1,1)$ -form on X corresponding to it. The hyperbolic form on \mathbb{H} is given by

$$\frac{\sqrt{-1}}{2} \cdot \frac{dz \wedge d\bar{z}}{\text{Im}(z)^2}.$$

So on X , the form $\mu_X^{\text{hyp}}(z)$ is given by

$$\mu_X^{\text{hyp}}(z) := \frac{\sqrt{-1}}{2} \cdot \frac{dz \wedge d\bar{z}}{\text{Im}(z)^2},$$

for $z \in X$. The total volume $\text{vol}_{\text{hyp}}(X)$ of X with respect to the hyperbolic metric μ_X^{hyp} is given by the formula

$$\text{vol}_{\text{hyp}}(X) = 4\pi(g-1).$$

Let

$$\mu_X^{\text{shyp}}(z) := \frac{\mu_X^{\text{hyp}}(z)}{\text{vol}_{\text{hyp}}(X)}$$

denote the rescaled hyperbolic metric on X , which is normalized in such a way that the volume of X is one.

2.2. Estimates of the Bergman kernel. For any $z \in X$, the Bergman kernel \mathcal{B}_X associated to the Riemann surface X is given by the following formula

$$\mathcal{B}_X(z) := \sum_{j=1}^g y^2 |f_j(z)|^2,$$

where $y = \text{Im } z$.

The injectivity radius r_X of X is defined as

$$r_X := \inf \{d_{\mathbb{H}}(z, \gamma z) \mid z \in \mathbb{H}, \gamma \in \Gamma \setminus \{\text{id}\}\},$$

where $d_{\mathbb{H}}(z, \gamma z)$ denotes the hyperbolic distance between z and γz .

Let f be any positive, smooth, real valued decreasing function defined on $\mathbb{R}_{\geq 0}$. From [9, Lemma 4], for any $\delta > r_X/2$, and assuming that all the involved integrals exist, we have the following inequality

$$\begin{aligned} \int_0^\infty f(\rho) dN_\Gamma(z_1, z_2; \rho) &\leq \int_0^\delta f(\rho) dN_\Gamma(z_1, z_2; \rho) \\ &+ f(\delta) \frac{\sinh(r_X/2) \sinh(\delta)}{\sinh^2(r_X/4)} + \frac{1}{2 \sinh^2(r_X/4)} \int_\delta^\infty f(\rho) \sinh(\rho + r_X/2) d\rho, \end{aligned} \quad (2.3)$$

where

$$N_\Gamma(z_1, z_2; \rho) := \text{card} \{ \gamma \mid \gamma \in \Gamma, d_{\mathbb{H}}(z_1, \gamma z_2) \leq \rho \}.$$

Notice that the above injectivity radius r_X is twice the injectivity radius defined in [9].

Theorem 2.1. *For any $z \in X$, the following estimate holds:*

$$\mathcal{B}_X(z) \leq B_X := \frac{48}{\pi} + \frac{4}{3\pi \sinh^2(r_X/4)}.$$

Proof. Substituting $k = 1$ in inequality (13) of [6], we arrive at

$$\mathcal{B}_X(z) \leq \frac{\sqrt{2}}{3\pi} \sum_{\gamma \in \Gamma} \frac{1}{\cosh^2(\rho_{\gamma,z}/2)} \int_{\rho_{\gamma,z}}^{\infty} \frac{ue^{-u/2}}{\sqrt{\cosh(u) - \cosh(\rho_{\gamma,z})}} du, \quad (2.4)$$

where $\rho_{\gamma,z} = d_{\mathbb{H}}(z, \gamma z)$. Using the fact that $u \leq \sinh(u)$ for all $u \geq 0$,

$$\begin{aligned} \int_{\rho_{\gamma,z}}^{\infty} \frac{ue^{-u/2}}{\sqrt{\cosh(u) - \cosh(\rho_{\gamma,z})}} du &\leq \int_{\rho_{\gamma,z}}^{\infty} \frac{ue^{-u/2}}{\sqrt{\cosh(u) - 1}} du \\ &= \int_{\rho_{\gamma,z}}^{\infty} \frac{ue^{-u/2}}{\sqrt{2 \sinh^2(u/2)}} du \leq \sqrt{2} \int_{\rho_{\gamma,z}}^{\infty} e^{-u/2} du = 2\sqrt{2}e^{-\rho_{\gamma,z}}. \end{aligned} \quad (2.5)$$

Combining (2.4) and (2.5), and using the fact that the inequality $\cosh(u) \geq e^u/2$ holds for all $u \geq 0$, it follows that

$$\mathcal{B}_X(z) \leq \frac{4}{3\pi} \sum_{\gamma \in \Gamma} \frac{e^{-\rho_{\gamma,z}}}{\cosh^2(\rho_{\gamma,z}/2)} \leq \frac{16}{3\pi} \sum_{\gamma \in \Gamma} \frac{e^{-\rho_{\gamma,z}}}{e^{\rho_{\gamma,z}}} = \frac{16}{3\pi} \int_0^{\infty} e^{-2\rho} dN_{\Gamma}(z, \gamma z; \rho).$$

As $e^{-2\rho}$ is a monotonically decreasing function in $\rho \in \mathbb{R}_{\geq 0}$, using (2.3) we compute

$$\begin{aligned} \mathcal{B}_X(z) &\leq \frac{16}{3\pi} \int_0^{\frac{3r_X}{4}} e^{-2\rho} dN_{\Gamma}(z, \gamma z; \rho) \\ &+ \frac{16e^{-\frac{3r_X}{2}} \sinh(r_X/2) \sinh(3r_X/4)}{3\pi \sinh^2(r_X/4)} + \frac{8}{3\pi \sinh^2(r_X/4)} \int_{\frac{3r_X}{4}}^{\infty} e^{-2\rho} \sinh\left(\rho + \frac{r_X}{2}\right) d\rho. \end{aligned} \quad (2.6)$$

From the definition of the injectivity radius r_X we have

$$\frac{16}{3\pi} \int_0^{\frac{3r_X}{4}} e^{-2\rho} dN_{\Gamma}(z, \gamma z; \rho) = \frac{16}{3\pi}. \quad (2.7)$$

Using the fact that $\sinh(u)$ is a monotone increasing function and that the inequality $\cosh(u) \leq e^u$ holds for all $u \geq 0$, we have the following estimate for the second term on the right-hand side of inequality in (2.6):

$$\begin{aligned} \frac{16e^{-\frac{3r_X}{2}} \sinh(r_X/2) \sinh(3r_X/4)}{3\pi \sinh^2(r_X/4)} &\leq \frac{16e^{-\frac{3r_X}{2}} \sinh(r_X/2) \sinh(r_X)}{3\pi \sinh^2(r_X/4)} \\ &\leq \frac{128e^{-\frac{3r_X}{2}} \cosh^2(r_X/4) \cosh(r_X/2)}{3\pi} \leq \frac{128e^{-\frac{r_X}{2}}}{3\pi} \leq \frac{128}{3\pi}. \end{aligned} \quad (2.8)$$

Using the fact that

$$\sinh(u) \leq e^u/2$$

for all $u \geq 0$, we derive the following estimate for the third term on the right-hand side of the inequality in (2.6):

$$\begin{aligned} &\frac{8}{3\pi \sinh^2(r_X/4)} \int_{\frac{3r_X}{4}}^{\infty} e^{-2\rho} \sinh\left(\rho + \frac{r_X}{2}\right) d\rho \\ &\leq \frac{4e^{\frac{r_X}{2}}}{3\pi \sinh^2(r_X/4)} \int_{\frac{3r_X}{4}}^{\infty} e^{-\rho} d\rho = \frac{4e^{-\frac{r_X}{4}}}{3\pi \sinh^2(r_X/4)} \leq \frac{4}{3\pi \sinh^2(r_X/4)}. \end{aligned} \quad (2.9)$$

Now the theorem follows from (2.7), (2.8), and (2.9). \square

Let $\{X_N\}_{N \in \mathcal{N}}$, indexed by $\mathcal{N} \subseteq \mathbb{N}$, be a set of compact hyperbolic Riemann surfaces. We say that the sequence is *admissible* if it is one of the following two types:

- (1) If $\mathcal{N} = \mathbb{N}$ and $N \in \mathcal{N}$, then X_{N+1} is a finite degree unramified cover of X_N .
- (2) Let $\mathcal{N} \subset \mathbb{N}$ be such that for each $N \in \mathcal{N}$, the modular curves $X_0(N)$, $X_1(N)$, $X(N)$, have genus $g > 1$. We consider families of modular curves $\{X_N\}_{N \in \mathcal{N}}$ given by

$$\{X_0(N)\}_{N \in \mathcal{N}}, \quad \{X_1(N)\}_{N \in \mathcal{N}}, \quad \{X(N)\}_{N \in \mathcal{N}}.$$

See [8, p. 695–696, Definition 5.1].

Let $q_{\mathcal{N}} \in \mathcal{N}$ be the minimal element of the indexing set \mathcal{N} . So in Case (1), we have $q_{\mathcal{N}} = 0$, while in Case (2), the integer $q_{\mathcal{N}}$ is the smallest prime in \mathcal{N} .

Corollary 2.2. *Let $\{X_N\}_{N \in \mathcal{N}}$ be an admissible sequence of compact hyperbolic Riemann surfaces. Then, for all $N \in \mathcal{N}$, the Bergman kernel $\mathcal{B}_{X_N}(z)$ is bounded by a constant which depends only on the Riemann surface $X_{q_{\mathcal{N}}}$.*

Proof. From Theorem 2.1, we have

$$\mathcal{B}_{X_N}(z) \leq B_{X_N} = O\left(\frac{1}{r_{X_N}^2}\right). \quad (2.10)$$

Recall that injectivity radius r_{X_N} is equal to ℓ_{X_N} , the length of the shortest geodesic on X_N . From assertion (a) in [8, Lemma 5.3] we know that for all $N \in \mathcal{N}$, the number $\frac{1}{r_{X_N}}$ is bounded by a number that depends only on the Riemann surface $X_{q_{\mathcal{N}}}$. Therefore, the estimate (2.10) completes the proof. \square

Remark 2.3. In [2], B.-Y. Chen and S. Fu have also derived a similar estimate for the Bergman kernel as in Corollary 2.2. However, their estimate is valid only for any compact hyperbolic Riemann surfaces with injectivity radius greater than or equal to $\log(3)$.

3. CARTESIAN PRODUCT X^d

In this section, we introduce the hyperbolic and canonical metrics defined over the d -fold Cartesian product X^d of X . We, then compute an estimate for the volume form associated to the canonical metric.

3.1. Canonical and hyperbolic metrics. Take X as before. Let X^d denote the d -fold Cartesian product $X \times \cdots \times X$. For each $1 \leq i \leq d$, let

$$p_i : X^d \longrightarrow X$$

be the projection to the i -th factor. Define

$$\mu_{X^d}^{\text{hyp}} = \sum_{i=1}^d p_i^* \mu_X^{\text{hyp}} \quad \text{and} \quad \mu_{X^d}^{\text{shyp}} = \sum_{i=1}^d p_i^* \mu_X^{\text{shyp}}.$$

We denote by $\mu_{X^d, \text{vol}}^{\text{shyp}}$ the volume form associated to $\mu_{X^d}^{\text{shyp}}$. Note that the total volume of X^d with respect to $\mu_{X^d, \text{vol}}^{\text{shyp}}$ is 1, because the total volume of X with respect to μ_X^{shyp} is 1.

With respect to a local coordinate $z = (z_1, \dots, z_d)$ on X^d , where $z_i = x_i + \sqrt{-1}y_i$ are hyperbolic coordinates on X , the hyperbolic volume form is given by

$$\mu_{X^d, \text{vol}}^{\text{shyp}}(z) = \frac{1}{(\text{vol}_{\text{hyp}}(X))^d} \bigwedge_{j=1}^d \frac{\sqrt{-1}}{2} \cdot \frac{dz_j \wedge d\bar{z}_j}{y_j^2} = \frac{1}{(4\pi(g-1))^d} \bigwedge_{j=1}^d \frac{\sqrt{-1}}{2} \cdot \frac{dz_j \wedge d\bar{z}_j}{y_j^2}.$$

The gonality of X is defined to be the smallest among all positive integers m such that X admits a nonconstant holomorphic map to \mathbb{CP}^1 of degree m . The gonality of X will be denoted by $\eta(X)$. So $\eta(X) = 2$ if and only if X is hyperelliptic.

We assume that $d < \eta(X)$.

Let $\text{Pic}^d(X)$ denote the component of the Picard group of X that parametrizes all the holomorphic line bundles of degree d . Consider the holomorphic map

$$\phi : X^d \longrightarrow \text{Pic}^d(X), \quad (x_1, \dots, x_d) \longmapsto \mathcal{O}_X(x_1 + \dots + x_d). \quad (3.1)$$

Since $d < m$, it can be shown that the fibers of the above map ϕ are zero dimensional. Indeed, if

$$\phi((x_1, \dots, x_d)) = \phi((y_1, \dots, y_d)),$$

the holomorphic line bundle $\mathcal{O}_X(x_1 + \dots + x_d)$ has two nonzero sections given by the two effective divisors $x_1 + \dots + x_d$ and $y_1 + \dots + y_d$. These two sections can't be linearly independent because that would contradict the assumption on d that it is strictly smaller than $\eta(X)$. Since two sections are constant multiples of each other, it follows that (x_1, \dots, x_d) and (y_1, \dots, y_d) differ by a permutation of $\{1, \dots, d\}$. Therefore, we have the following:

Lemma 3.1. *Any two points of X^d lying in a fiber of the map ϕ differ by a permutation of $\{1, \dots, d\}$.*

The variety $\text{Pic}^d(X)$ is a torsor for $\text{Jac}(X)$, because any two holomorphic line bundles of degree d differ by tensoring with a unique holomorphic line bundle of degree zero. Therefore, by fixing a point of $\text{Pic}^d(X)$ we may identify $\text{Jac}(X)$ with $\text{Pic}^d(X)$. Using this identification, we get a Kähler metric on $\text{Pic}^d(X)$ given by the metric on $\text{Jac}(X)$ constructed in (2.1). This metric on $\text{Pic}^d(X)$ will be denoted by g_d . We note that g_d does not depend on the choice of the point in $\text{Pic}^d(X)$ used in identifying $\text{Jac}(X)$ with $\text{Pic}^d(X)$.

The pullback ϕ^*g_d is the canonical metric on X^d , which we denote by $\mu_{X^d}^{\text{can}}$. The canonical metric degenerates along the divisor where two or more coordinates coincide (where the action of the group of permutations of $\{1, \dots, d\}$ is not free). In Remark 4.2 we will see that this is precisely the locus where $\mu_{X^d}^{\text{can}}$ degenerates.

As in Section 2.1, let $\{f_1, \dots, f_g\}$ be an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. The (1,1)-form associated to the canonical metric $\mu_{X^d}^{\text{can}}$ is given by

$$\mu_{X^d}^{\text{can}} = \frac{\sqrt{-1}}{2g^d} \sum_{j=1}^g \sum_{a,b=1}^d f_j(z_a) \overline{f_j(z_b)} dz_a \wedge d\bar{z}_b. \quad (3.2)$$

The volume form associated to the canonical metric $\mu_{X^d}^{\text{can}}$ measures the total volume of X^d to be one.

For any $z = (z_1, \dots, z_d) \in X^d$, the Bergman kernel associated to X^d is given by the formula

$$\mathcal{B}_{X^d}(z) = \prod_{i=1}^d \mathcal{B}_X(z_i, w_i).$$

3.2. Estimates of $\mu_{X^d, \text{vol}}^{\text{can}}$. In this subsection, using the estimate for the Bergman kernel $\mathcal{B}_X(z)$ derived in Theorem 2.1, we estimate $\mu_{X^d, \text{vol}}^{\text{can}}$, the volume form associated to the canonical metric $\mu_{X^d}^{\text{can}}$.

Theorem 3.2. *For any $z \in X^d$, the following inequality holds:*

$$\left| \frac{\mu_{X^d, \text{vol}}^{\text{can}}(z)}{\mu_{X^d, \text{vol}}^{\text{shyp}}(z)} \right| \leq (d!)^2 \left(\frac{\text{vol}_{\text{hyp}}(X) B_X}{g^{d-1}} \right)^d.$$

Proof. For any $z = (z_1, \dots, z_d) \in X^d$, the canonical volume form $\mu_{X^d, \text{vol}}^{\text{can}}$ is given by

$$\begin{aligned} \mu_{X^d, \text{vol}}^{\text{can}}(z) &= \left(\frac{\sqrt{-1}}{2g^d} \right)^d \sum_{\substack{j_1, \dots, j_d \in \{1, \dots, g\} \\ \sigma, \tau \in S_d}} f_{j_1}(z_{\sigma(1)}) \overline{f_{j_1}(z_{\tau(1)})} \cdots f_{j_d}(z_{\sigma(d)}) \overline{f_{j_d}(z_{\tau(d)})} \bigwedge_{k=1}^d dz_{\sigma(k)} \wedge d\bar{z}_{\tau(k)} = \\ &= \left(\frac{\sqrt{-1}}{2g^d} \right)^d \sum_{\substack{j_1, \dots, j_d \in \{1, \dots, g\} \\ \sigma, \tau \in S_d}} \text{sgn}(\sigma) \text{sgn}(\tau) f_{j_1}(z_{\sigma(1)}) \overline{f_{j_1}(z_{\tau(1)})} \cdots f_{j_d}(z_{\sigma(d)}) \overline{f_{j_d}(z_{\tau(d)})} \bigwedge_{k=1}^d dz_k \wedge d\bar{z}_k. \end{aligned}$$

Using the above expression, we observe that

$$\begin{aligned} \left| \frac{\mu_{X^d, \text{vol}}^{\text{can}}(z)}{\mu_{X^d, \text{vol}}^{\text{shyp}}(z)} \right|^2 &= \left(\frac{\text{vol}_{\text{hyp}}(X)}{g^d} \right)^{2d} \\ &\times \left| \left(\prod_{k=1}^d y_k^2 \right) \cdot \sum_{\substack{j_1, \dots, j_d \in \{1, \dots, g\} \\ \sigma, \tau \in S_d}} \text{sgn}(\sigma) \text{sgn}(\tau) f_{j_1}(z_{\sigma(1)}) \overline{f_{j_1}(z_{\tau(1)})} \cdots f_{j_d}(z_{\sigma(d)}) \overline{f_{j_d}(z_{\tau(d)})} \right|^2. \end{aligned}$$

Since the number of terms in the above summation are $(d!)^2 g^d$, we arrive at the inequality

$$\begin{aligned} \left| \frac{\mu_{X^d, \text{vol}}^{\text{can}}(z)}{\mu_{X^d, \text{vol}}^{\text{shyp}}(z)} \right|^2 &\leq (d!)^4 \left(\frac{g \text{vol}_{\text{hyp}}(X)}{g^d} \right)^{2d} \\ &\times \sup_{\substack{j_1, \dots, j_d \in \{1, \dots, g\} \\ \sigma, \tau \in S_d, z \in X^d}} \left| \left(\prod_{k=1}^d y_k^2 \right) \cdot f_{j_1}(z_{\sigma(1)}) \overline{f_{j_1}(z_{\tau(1)})} \cdots f_{j_d}(z_{\sigma(d)}) \overline{f_{j_d}(z_{\tau(d)})} \right|^2. \end{aligned} \quad (3.3)$$

From Theorem 2.1, we derive

$$\begin{aligned} &\sup_{\substack{j_1, \dots, j_d \in \{1, \dots, g\} \\ \sigma, \tau \in S_d, z \in X^d}} \left| \left(\prod_{k=1}^d y_k^2 \right) \cdot f_{j_1}(z_{\sigma(1)}) \overline{f_{j_1}(z_{\tau(1)})} \cdots f_{j_d}(z_{\sigma(d)}) \overline{f_{j_d}(z_{\tau(d)})} \right|^2 \\ &\leq \sup_{z \in X^d} (\mathcal{B}_{X^d}(z))^2 \leq (B_X)^{2d}. \end{aligned} \quad (3.4)$$

Combining the inequalities (3.3) and (3.4), the proof is completed. \square

4. SINGULARITIES OF THE CANONICAL METRIC ON THE SYMMETRIC PRODUCT

As before, take $d < \eta(X)$. Let S_d denote the group permutation of $\{1, \dots, d\}$. It acts on X^d by permuting the factors. Let $\text{Sym}^d(X)$ denote the d -fold symmetric product of X . In other words, $\text{Sym}^d(X)$ is the quotient of X^d for the action of S_d .

The metric $\mu_{\text{Sym}^d(X)}^{\text{can}}$ on X^d is clearly invariant under the action of the group S_d . Let us denote the push-forward of the canonical metric $\mu_{X^d}^{\text{can}}$ onto $\text{Sym}^d(X)$.

Proposition 4.1. *Consider the map $\phi : X^d \rightarrow \text{Pic}^d(X)$ in (3.1). It factors through the quotient $X^d \rightarrow X^d/S_d = \text{Sym}^d(X)$. The resulting map*

$$\text{Sym}^d(X) \rightarrow \text{Pic}^d(X)$$

is an embedding.

Proof. If two elements (x_1, \dots, x_d) and (y_1, \dots, y_d) of X^d lie in the same orbit for the action of S_d on X^d , then the line bundles $\mathcal{O}_X(x_1 + \dots + x_d)$ and $\mathcal{O}_X(y_1 + \dots + y_d)$ are isomorphic. Hence ϕ descends to a morphism

$$\varphi : \text{Sym}^d(X) \rightarrow \text{Pic}^d(X). \quad (4.1)$$

From Lemma 3.1 we know that φ is injective. Therefore, it suffices to show that φ is an immersion.

Take any point $\underline{x} = \{x_1, \dots, x_d\} \in \text{Sym}^d(X)$. The divisor $\sum_{i=1}^d x_i$ will be denoted by D . Let

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow Q'(\underline{x}) := \mathcal{O}_X/\mathcal{O}_X(-D) \rightarrow 0$$

be the short exact sequence corresponding to the point \underline{x} . Tensoring it with the line bundle $\mathcal{O}_X(-D)^* = \mathcal{O}_X(D)$ we get the short exact sequence

$$\begin{aligned} 0 \rightarrow \text{End}(\mathcal{O}_X(-D)) = \mathcal{O}_X &\rightarrow \text{Hom}(\mathcal{O}_X(-D), \mathcal{O}_X) = \mathcal{O}_X(D) \\ &\rightarrow Q(\underline{x}) := \text{Hom}(\mathcal{O}_X(-D), Q'(\underline{x})) \rightarrow 0. \end{aligned}$$

Let

$$0 \rightarrow H^0(X, \mathcal{O}_X) \xrightarrow{\alpha} H^0(X, \mathcal{O}_X(D)) \xrightarrow{\beta} H^0(X, Q(\underline{x})) \xrightarrow{\gamma} H^1(X, \mathcal{O}_X) \quad (4.2)$$

be the long exact sequence of cohomologies associated to this short exact sequence of sheaves.

The holomorphic tangent space to $\text{Sym}^d(X)$ at \underline{x} is

$$T_{\underline{x}} \text{Sym}^d(X) = H^0(X, Q(\underline{x})),$$

and the tangent bundle of $\text{Pic}^d(X)$ is the trivial vector bundle with fiber $H^1(X, \mathcal{O}_X)$. The differential at \underline{x} of the map φ in (4.1)

$$(d\varphi)(\underline{x}) : T_{\underline{x}} \text{Sym}^d(X) = H^0(X, Q(\underline{x})) \rightarrow T_{\varphi(\underline{x})} \text{Pic}^d(X) = H^1(X, \mathcal{O}_X)$$

satisfies the identity

$$(d\varphi)(\underline{x}) = \gamma, \quad (4.3)$$

where γ is the homomorphism in (4.2).

Now, $H^0(X, \mathcal{O}_X) = \mathbb{C}$. In the proof of Lemma 3.1 we saw that

$$H^0(X, \mathcal{O}_X(D)) = \mathbb{C}.$$

Hence the homomorphism α in (4.2) is an isomorphism. Consequently, β in the exact sequence (4.2) is the zero homomorphism and γ in (4.2) is injective.

Since γ in (4.2) is injective, from (4.3) we conclude that φ is an immersion. \square

Remark 4.2. Since φ is an embedding, the metric $\mu_{\text{Sym}^d(X)}^{\text{can}}$ on $\text{Sym}^d(X)$ is nonsingular. Therefore, the metric $\mu_{X^d}^{\text{can}}$ on X^d is singular exactly on the divisor where the quotient map $X^d \rightarrow \text{Sym}^d(X)$ is ramified. We note that this ramification divisor consists of all points of X^d such that the d points of X are not distinct.

5. AUTOMORPHISMS OF $\text{Sym}^d(X)$

Consider the nonsingular Kähler metric $\mu_{\text{Sym}^d(X)}^{\text{can}}$ on $\text{Sym}^d(X)$ (see Remark 4.2).

Theorem 5.1. *Let $T : \text{Sym}^d(X) \rightarrow \text{Sym}^d(X)$ be any holomorphic automorphism. Then the pulled back Kähler form $T^* \mu_{\text{Sym}^d(X)}^{\text{can}}$ coincides with $\mu_{\text{Sym}^d(X)}^{\text{can}}$. In particular, T is a isometry for the metric $\mu_{\text{Sym}^d(X)}^{\text{can}}$.*

Proof. Since φ (constructed in (4.1)) is the Albanese map for $\text{Sym}^d(X)$, there is a holomorphic automorphism

$$\widehat{T} : \text{Pic}^d(X) \rightarrow \text{Pic}^d(X)$$

such that

$$\varphi \circ T = \widehat{T} \circ \varphi. \quad (5.1)$$

From [5] we know that \widehat{T} preserves the polarization on $\text{Pic}^d(X)$. A theorem due to Weil says a holomorphic automorphism of $\text{Jac}(X) = \text{Pic}^0(X)$ that preserves the polarization is generated by the following:

- translations of $\text{Pic}^0(X)$,
- automorphisms of $\text{Pic}^0(X)$ given by the holomorphic automorphisms of X , and
- the inversion of $\text{Pic}^0(X)$ defined by $L \mapsto L^*$.

(See [16, Hauptsatz, p. 35].) But all these three types of automorphisms of $\text{Pic}^0(X)$ are isometries for the flat Kähler form on $\text{Pic}^0(X)$ constructed in (2.1). From this it follows immediately that \widehat{T} is an isometry for the flat Kähler form g_d on $\text{Pic}^d(X)$ constructed in Section 3.1. Since \widehat{T} is an isometry, from (5.1) it follows immediately that $T^* \omega_d = \omega_d$. \square

ACKNOWLEDGEMENTS

We thank the referee for pointing out a reference. The second-named author wishes to thank the University of Hyderabad for hospitality while the work was carried out. He is supported by a J. C. Bose Fellowship.

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